The Physics of Juggling

By Bengt Magnusson and Bruce Tiemann

Juggling, the art of controlling more objects up in the air than you have hands, has amazed and entertained people for thousands of years. From ancient Egyptian hieroglyphics through old Japanese woodcuts to present day photographs, we have images and stories of people manipulating stunning numbers of objects. A nameless Egyptian 4000 years ago was said to be able to juggle nine balls. Two Japanese jugglers, both living many hundred years ago, have also left their marks in the history of juggling. One was said to have stopped a war with his nine-ball juggling. His enemies fled in panic before his supposed magic since nobody but a powerful sorcerer could possibly perform such a feat. The other could juggle seven swords. Early in our own century, Jenny Yaeger and Enrico Rastelli set records by becoming the only people ever to verifiably juggle ten balls each. That number was surpassed recently, but with an easier juggling prop, when Sergei Ignatov and Albert Petrovsky successfully juggled eleven rings each. Yet a juggler does not need large numbers of objects to spellbind an audience, as shown by present-day performers such as Michael Moshen and the Airjazz trio. Three-ball juggling has enough potential to keep a creative juggler busy for a lifetime.

Many people have asked themselves how jugglers can perform the tricks they do. How is it possible to control all the objects in the air? A first step towards an answer to this question would be to explore the basic physical laws that govern the activity. Being both physicists and active jugglers, we set out to discover some of these laws almost three years ago during our senior year at the California Institute of Technology. While the physics involved does not go beyond the level of a first-term freshman, this exercise of ours has been very entertaining and also an opportunity for "playing by the seashore."

The fundamental observation, familiar to any juggler, is that all basic juggling is performed in a very strong pattern, where each hand does exactly the same thing. The basic pattern, which is the same no matter what kind of objects are being juggled, comes in two versions, one used when an odd number of objects are juggled, and one used for an even number. For odd numbers, each hand throws the object in an arc across to the other hand, where it is caught and thrown back in a similar arc. The object is released close to the center of the pattern and is caught on the outside. The two arcs intersect in front of the juggler, and each object visits both hands. This pattern is called the *cascade* (Fig. 1). For even numbers, each hand throws the object from the inside, and the same hand catches it on the outside. The two arcs do not intersect, and the objects never switch hands. This pattern is called the *fountain* (Fig. 2). Most nonjugglers have the misconception that all juggling takes place in a "circular" pattern. This variation is called the *shower* (Fig. 3) and is considerably harder than the patterns described above. In a shower, there is only one arc instead of two, and as a result there is much less time for each throw or the need for a much higher arc, both of which make the juggling harder. A pattern intermediate between the cascade and the shower, called the half-shower, is also fairly common.

To derive the relationship between throw height and number of balls in the basic patterns, we need to introduce a few variables. Let the time between consecutive throws from one hand be τ. The object will always spend some time in the hand: it is caught, carried over to the throwing position, and released. Call this time $\theta\tau$. θ is then the fractional dwell time, and $0 < \theta < 1$. Let n be the number of objects juggled, and h be the height of each throw.

First assume n is even. Each hand then has $\frac{1}{2} n$ objects, and it would take $\frac{1}{2} n\tau$ seconds for a hand to go through a full cycle. However, because of the dwell time, the first object must land a time $\theta\tau$ earlier, and the time t each object spends up in the air is $t = \frac{1}{2} n\tau - \theta\tau = \frac{1}{2}\tau(n - 2\theta)$. This formula holds true for odd n as well, although the derivation is somewhat different. Since $h = \frac{1}{8} gt^2$ we have, with $t = \frac{1}{2}\tau(n - 2\theta)$, that

$$h = \frac{1}{32} g(n - 2\theta)^2 \tau^2 \qquad (1)$$

Bengt Magnusson *received his B.S. in physics from the California Institute of Technology in 1987 and is currently a third-year graduate student at the University of California at Santa Barbara. His field of research is experimental particle astrophysics. He has been juggling a little over three years and has reached the nine-ball/five-club level.*

Bruce Tiemann *received his B.S. in chemistry from the California Institute of Technology in 1987 and is currently working with nonlinear optical compounds at the Jet Propulsion Laboratory in Pasadena. He has been juggling for about ten years and performs "reasonably well" with seven clubs and eight balls. Other hobbies include building lasers and amateur radio.*

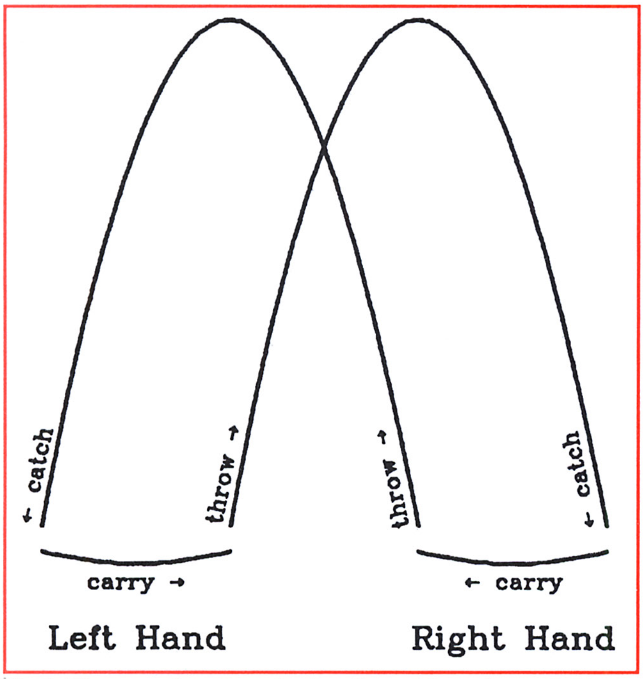

Fig. 1. The *cascade* pattern for juggling odd numbers of objects.

$n - 2\theta$ is the time average of the number of objects in the air. For the vast majority of jugglers, θ is within a few percent of 0.5. The reason for this is that $\theta = 0.5$ establishes a very strong rhythm: one hand will be making a throw exactly when the other hand is making a catch. τ varies from about 0.2 s to 0.8 s; for most jugglers $\tau \approx 0.5$ s.

τ and h are the limiting factors for maintaining a juggle. h determines how accurately you must aim; a high throw

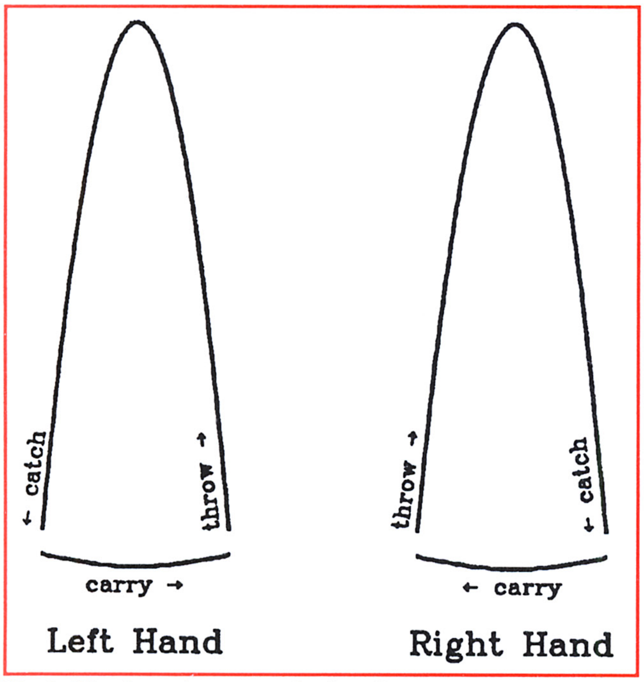

Fig. 2. The *fountain* pattern for juggling even numbers of objects.

must be aimed better than a low throw, and τ determines how fast you must be able to aim that well. It is easier to aim well if you have plenty of time to do it, much less so if you are rushed. Assume for simplicity that the objects are point-like. This removes the problem of analyzing collisions between objects, which is a very difficult problem to solve, especially with large rotating props like clubs. Let the width of each arc be s, with an error margin of Δs on either side. Let the object be released with velocity v_0 at an angle α from vertical, with an error $\Delta\alpha$ that gives rise to the error in width Δs (Fig. 4). To reach height h we must have $v_{0z} = (2gh)^{1/2}$. Then $v_{0x}/v_{0z} = \tan\alpha$, giving $v_{0x} = v_{0z}\tan\alpha$ and $s = t\,v_{0x} = (8h/g)^{1/2} v_{0z}\tan\alpha = (8h/g)^{1/2}(2gh)^{1/2}\tan\alpha = 4h\tan\alpha$, or $\alpha = \arctan(s/4h)$. Now $\alpha + \Delta\alpha = \arctan[(s + \Delta s)/4h]$, giving $\Delta\alpha = \arctan[(s + \Delta s)/4h] - \arctan(s/4h)$. Since $\arctan(x) = x +$ terms of order x^3 and higher, we have

$$\Delta\alpha \approx \frac{s + \Delta s}{4h} - \frac{s}{4h} = \frac{\Delta s}{4h} \qquad (2)$$

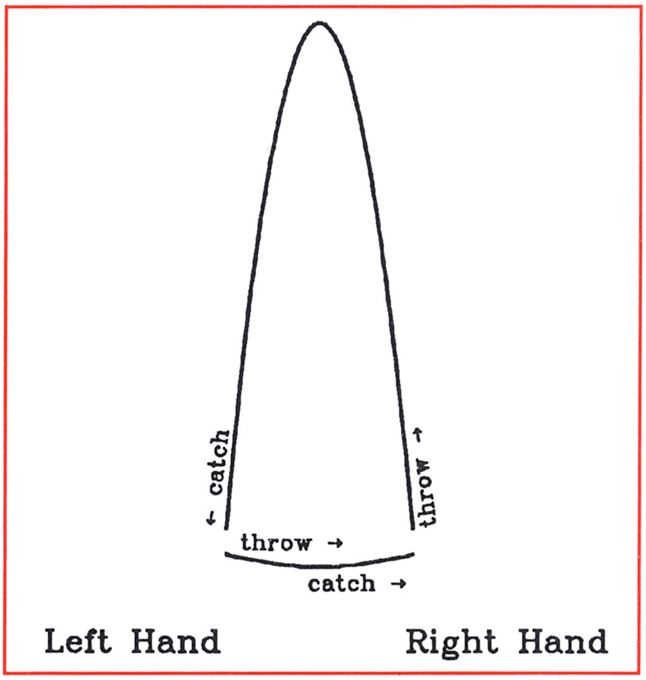

Fig. 3. The *shower* has only one arc and is more difficult to perform.

Δs is about 0.3 m for most jugglers, and the highest patterns juggled by the best jugglers with any consistency are about 6-m high, like Ignatov's 11 rings. This gives $\Delta\alpha = 0.6°$! Thus Ignatov had to aim each ring to within 0.6° just to be able to reach it. In order to avoid collisions, the aim would have to be even better. With $\alpha = \arctan(s/4h) \approx s/4h$, we can form $\Delta\alpha/\alpha \approx (\Delta s/4h)/(s/4h) = \Delta s/s$, i.e., the fractional error in the angle is independent of throw height and, with s being about 0.9 m, equal to ⅓ for most jugglers.

Since $\Delta\alpha$ is limiting the throw height, it, together with

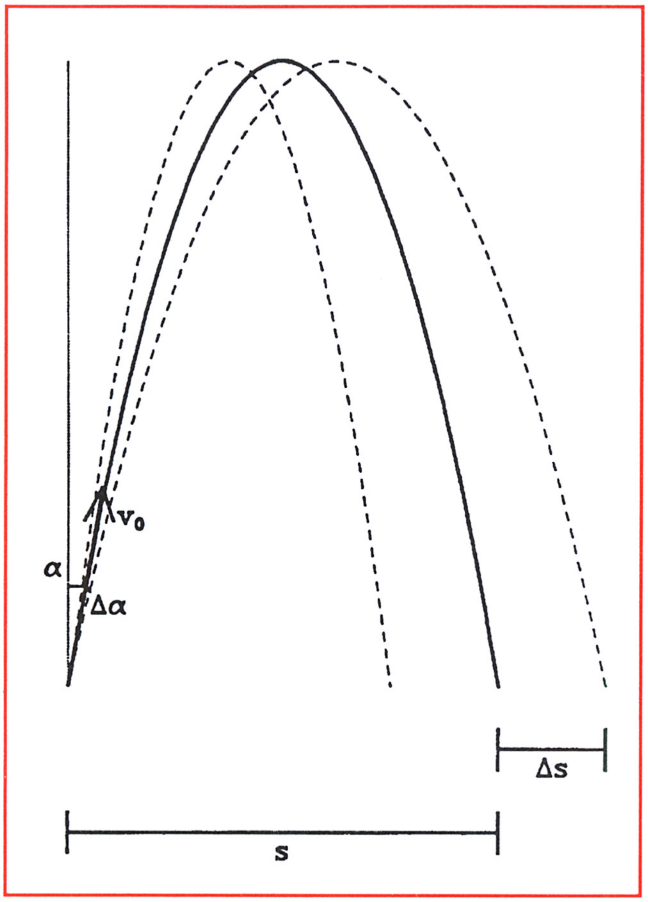

Fig. 4. Error geometry for a throw.

τ, limits the number of balls that can be juggled. Increasing θ will decrease h somewhat, thus increasing $\Delta\alpha$, but the rhythm of the pattern will be lost, and the time each hand has to reach for the next incoming object, $\tau(1 - \theta)$, decreases. If a typical juggler, with τ = 0.5 s and θ = 0.5, wants to juggle seven balls, each throw would, according to Eq. (1), have to be about 2.8-m high. If the same juggler moved to the moon, where g is 1/6 of its value at earth, and keeps the same τ, θ, and h, we can solve for n to find out how many balls he could juggle there. $n = 2\theta + \tau^{-1}\left(32\,h/g_{moon}\right)^{1/2} \approx 15.7$ or 15 balls. Fifteen is still small enough to allow all of the balls to fit in the pattern without causing constant collisions. It could be argued that the sheer number of balls up in the air would be too staggeringly complex for the brain to process, thus making 15 balls impossible. In our experience, that is not the case. The brain learns the pattern recognition for a higher number in a remarkably short period of time, much sooner than the needed motoric skills are developed. Even the problem of holding 15 balls in two hands at the start of the juggle and then releasing them in a controlled pattern is not insurmountable. One could, if necessary, use a juggling machine to start the pattern. We therefore believe 15 balls to be a very reasonable number to be juggled on the moon. Today's best Earth-bound jugglers can control 9 balls at a time; on the moon, they should then be able to do about 19 balls!

If you are juggling n objects, you may wonder at what height you would juggle a different number m, keeping τ, θ constant. The immediate answer from most people is that twice as many objects would require a pattern four times as high, but they forget the effect of θ. The correct result is, from Eq. (1),

$$\frac{h_n}{h_m} = \left(\frac{n - 2\theta}{m - 2\theta}\right)^2 \qquad (3)$$

With θ = 0.5 we get, for example, f_6/f_3 = 6.25, which is significantly different from four. It turns out that most jugglers, rather than accepting the smaller $\Delta\alpha$ of a higher h, accept a somewhat smaller τ and increase h somewhat less than might be expected.

In passing, it can be mentioned that throws of varying heights can be combined in a single juggling pattern, provided some simple rules are followed. These combinations lead to an entire class of juggling tricks we call "site-swaps." First we must introduce some notation, originally invented by Paul Klymack from Santa Cruz, who independently of us discovered a subset of these tricks. In this notation, the number n denotes the path into which you would throw the ball if you were juggling that many of them. Thus, a 3 means you throw the ball in a low arc across to the other hand, a 4 is a little higher but to the same hand, a 5 much higher across to the other hand, etc. A 0 denotes an empty hand, a 1 is a quick hand-off to the other hand, and a 2 means you just hold on to the ball for a beat. If you now put together a string of numbers, like 7 5 6 2, it is sometimes possible for you to juggle this pattern without collisions or inconsistencies. Your hands always take an action at evenly spaced intervals and throw the balls according to the string of numbers. The balls will go to widely varying heights, and the pattern may look very confusing, but if you picked your string of numbers correctly, they will always land, as if by magic, right where you need them, right when you need them.

What are the criteria for a correct string of numbers? First of all, the numbers must average to the number of balls you are trying to juggle. 7 5 6 2, for example, averages to five, so that would be a five-ball trick. However, this is not enough. You must also make sure two balls will never land in the same hand at the same time. Take the 7 5 6 2 again. Since this is a five-ball trick, we can normalize the numbers to five, and call it a 2 0 1 -3. (The numbers now add up to zero.) The 2 will cause the ball thrown in position one to land two positions later, in position three. The ball in position three is moved forward one position, to four. The ball in position four is moved to three positions earlier, to position one. (The ball in position one has already been moved out of the way.) Finally, the ball in position two stays there. We see that after one cycle, no two balls tried to be in the same position. Compare this to an illegal trick, like the 8 5 4 3. These also average to five and transcribe to 3 0 -1 -2. Here, the 0, -1, and -2 try to arrive in the second position, thus giving rise to a triple collision. The best way to discover the legal tricks is to have them generated by a computer. In doing that, we generated several thousand tricks, 99.9 percent of which were completely unknown to other jugglers before. If you are a juggler, and want to amaze and confuse your other juggling friends, try some of these tricks: 4 4 1 and 5 1 4 1 4 with three balls, 7 5 3 1 and 8 5 2 4 1 with four balls, and 6 6 6 6 1, 7 5 6 2, 7 5 7 5 1, and 9 5 5 5 1 with five balls. The mathematically interested reader may notice that the set of all site-swaps of length n (together with the operation composition of functions) forms a group isomorphic to sym (n).

The results above apply to all the common juggling props—balls, rings, and clubs. Clubs introduce an entirely different variable to the analysis. They rotate in flight, and must be caught by the handles. (Rings also rotate in flight, but they can be caught no matter what the rotation is.) This complication, together with their larger size, makes clubs by far the most difficult prop to juggle, and the official world record is only seven. The analysis of club spin produced a quite surprising result. By tossing the club higher, it will spin more since it is in the air for a longer time, and it seems natural that a double spin would have to be thrown four times as high as a single spin. It turns out that when you throw the club higher, you also make it spin faster, and a double spin is exactly twice as high as a single spin.

A good model of how a club is thrown is as follows. The club is thrown with the juggler's arm acting as a pivot arm of length ℓ from his elbow to the center of mass of the club, rotating around a fixed point (his elbow) (Fig. 5). Let ℓ_1 be the distance from the juggler's elbow to the tip of the club handle. If the juggler uses a lot of wrist action when throwing the club (this varies from juggler to juggler), ℓ may not exactly equal the distance from the juggler's elbow to the center of mass of the club, but there will be some effective lever arm ℓ_{eff}, which is quite constant for any individual juggler. The same holds true for ℓ_1. Let ω be the angular velocity of the pivot arm. The center of mass of the club is released with velocity $v_0 = \omega\ell$ and stays in the air for a time $t = 2v_0/g = 2\omega\ell/g$ s. The tip of the handle starts with velocity $v_1 = \omega(\ell-\ell_1)$. In the center of mass frame, the tip of the handle has speed $v_0 - v_1 = \omega\ell - \omega(\ell-\ell_1)$, and this speed remains constant for the entire throw. We see that the handle rotates with the same angular frequency ω as the pivot arm had. Since the handle has to rotate by an angle of $2\pi m$ for m revolutions, it must stay up a time $t_m = 2\pi m/\omega$. This time must equal the time previously calculated: $2\pi m/\omega = 2\omega\ell/g$, or $\omega = (m\pi g/\ell)^{1/2}$. The throw height is then

$$h_m = \frac{1}{8}gt_m^2 = \frac{1}{8}g\left(\frac{2\pi m}{\omega}\right)^2 = \frac{m\pi\ell}{2} \tag{4}$$

Note how h_m is a linear function of m. Another noteworthy feature of this formula is its independence of g. The reason is that in a lower g, you must release the club slower

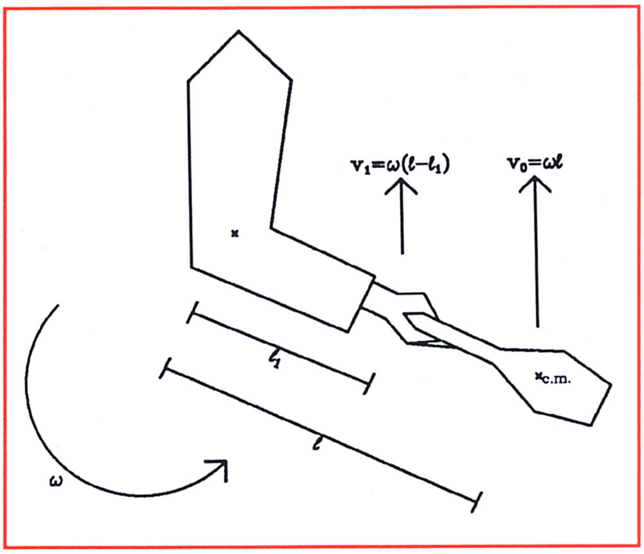

Fig. 5. Kinematics at a club launch.

if you want it to go up only to the same height as in a higher g. By doing that, you automatically put less spin on the club, but since g is lower, the club stays up in the air longer by the exact amount to compensate for the slower spin.

If you ever see a good five-club juggler, we would suggest you strobe your eyes by blinking hard and fast while looking at the pattern. Every once in a while you will notice how all clubs line up in parallel. It will only happen when an odd number of clubs are being juggled. Even numbers of clubs will have other, less-characteristic configurations. Why is this? Let n = number of clubs, m = number of spins, and ω, τ, and θ as before. Since each hand throws every τ seconds, the time between a left-hand throw and the following right-hand throw is $\frac{1}{2}\tau$. The angular separation between two clubs at the time of release is then $\beta = \frac{1}{2}\omega\tau$. With $h_m = m\pi\ell/2 = \frac{1}{32}g(n-2\theta)^2\tau^2$ we get $\ell = g(n-2\theta)^2\tau^2/16m\pi$, and $\omega = (m\pi g/\ell)^{1/2} = 4m\pi/(n-2\theta)\tau$, giving

$$\beta = \frac{\omega\tau}{2} = \frac{2m\pi}{n-2\theta}rad = \frac{360\,m}{n-2\theta}deg. \tag{5}$$

Thus we see that parallel clubs, $\beta = 180°$, occur only at $n = 2(m+\theta)$. n must be an integer, so θ must be either 0, 0.5, or 1. But $\theta = 0$ or 1 is nonphysical, which leaves us with $\theta = 0.5$, and $n = 2m + 1$. Since most jugglers have θ very close to 0.5, and since three clubs are usually juggled with single spins, five with doubles, seven with triples, etc., parallel clubs are frequently attained by jugglers. Parallel clubs could also happen at $\beta = 360°$, five clubs with quadruple spins, for example, but very few people can juggle this pattern. Other interesting angles, all at $\theta = 0.5$, are: 90°, five singles; 120°, four singles and seven doubles; 240°, four doubles and seven quads; and 270°, five triples. We have managed to capture most of these angles on photographs, which has given us a lot of confidence in our simple model for club juggling. The photograph shows the

Fig. 6. The authors juggle four and five clubs.

authors juggling four and five clubs with double spins, demonstrating $\beta = 240°$ and 180°, respectively (Fig. 6).

We would like to thank all of our juggling friends, who patiently performed for our cameras and stop watches to help us confirm our theories. In particular we thank Dan Bennett, David Deeble, Barry Friedman, Daniel Holzman, Tyler Linkin, Bob Mendelsohn, and Owen Morse. We have seen how juggling patterns obey fairly simple physical rules. Nowadays we don't run away in terror when we see skilled jugglers, but even though the physical background of juggling has been mapped out, the biological questions of how the juggler actually goes about utilizing these rules and acquiring the motoric skills remain unanswered. ♦